Equivalence of various pseudopotential approaches for Einstein-Maxwell fields

1982 J. Phys. A: Math. Gen. 15 2201

(http://iopscience.iop.org/0305-4470/15/7/029)

J. Phys. A: Math. Gen. 15 (1982) 2201–2207. Printed in Great Britain

Equivalence of various pseudopotential approaches for Einstein–Maxwell fields

Dietrich Kramer

Sektion Physik, Friedrich-Schiller-Universität Jena, DDR-6900 Jena, Max-Wien-Platz 1, GDR

Received 4 November 1981

Abstract. In the literature, various systems of linear eigenvalue equations from which the Einstein–Maxwell equations for stationary axisymmetric exterior fields follow as the integrability conditions were derived. In the present paper, these linear systems are shown to be equivalent; the explicit transformations mapping one form to another are given.

1. Introduction

Some (systems of) nonlinear differential equations have the remarkable property that they can be obtained as the integrability conditions of appropriate linear equations including a spectral parameter, t. Frequently it is possible to find special solutions to these linear eigenvalue equations. For instance, the assumption that only (simple) poles in the complex t plane occur leads to the so-called soliton solutions.

For the Einstein equations for stationary axisymmetric vacuum fields, soliton solutions have been given independently by several authors (Neugebauer 1979, 1980, Belinsky and Zakharov 1978, 1979, Harrison 1978). Following a series of papers by Kinnersley (1977) and Kinnersley and Chitre (1977, 1978a, b), Hauser and Ernst (1979, 1980) reformulated the problem of constructing finite elements of an internal symmetry group, in terms of a homogeneous Hilbert problem with respect to a linear integral equation. Cosgrove (1981) has shown that the soliton solutions are included in the Hauser–Ernst formalism.

Hauser and Ernst (1979, 1980) also generalised their method to include electromagnetic fields. Aleksejev (1980a) constructed N-soliton solutions of the Einstein–Maxwell equations. In all these investigations linear equations which imply the nonlinear field equations play an important role. Different authors introduced different (pseudo-) potentials: the Ω potentials (§ 2, Kramer and Neugebauer 1981), the F potentials (§ 3, Hauser and Ernst 1980, Jones 1980, Kinnersley and Chitre 1977, 1978a, b), and the Ψ potentials (§ 5, Aleksejev 1980a). It is the purpose of this paper to reveal the relationships between these quantities, and the equivalence of the corresponding linear eigenvalue equations. In particular, it turns out that the electrovac solution-generating transformation given by Cosgrove (1981) is contained in the result of Aleksejev (1980a), for $N = 1$.

The space–time metric is taken in the standard form

$$\begin{aligned} ds^2 &= f^{-1}[e^{2k}(d\rho^2+dz^2)+\rho^2\, d\varphi^2]-f(d\tau-\omega\, d\varphi)^2 \\ &= f^{-1}\, e^{2k}\, d\zeta\, d\bar{\zeta}+f_{AB}\, dx^A\, dx^B \qquad A, B=1,2 \end{aligned} \tag{1}$$

where f, k, and ω are independent of the coordinates $x^A=(\tau, \varphi)$. Throughout this paper a bar denotes complex conjugation.

2. The Ω potentials

By means of the Wahlquist–Estabrook (1975) method, Kramer and Neugebauer (1981) derived linear equations which imply the Ernst equations for stationary axisymmetric Einstein–Maxwell fields. In matrix notation, these linear equations can be written in the form

$$\Omega_{,\zeta}=(X_1+\lambda Y_1)\Omega \qquad \Omega_{,\bar{\zeta}}=(X_2+\lambda^{-1}Y_2)\Omega \tag{2}$$

where

$$\lambda=\left(\frac{1-2it\bar{\zeta}}{1+2it\zeta}\right)^{1/2} \qquad t=\text{constant} \tag{3}$$

and the 3×3 matrices X_α, Y_α ($\alpha=1,2$) are given by

$$X_\alpha=\begin{pmatrix} B_\alpha & 0 & E_\alpha \\ 0 & A_\alpha & 0 \\ -D_\alpha & 0 & \frac{1}{2}(A_\alpha+B_\alpha)\end{pmatrix} \qquad Y_\alpha=\begin{pmatrix} 0 & B_\alpha & 0 \\ A_\alpha & 0 & -E_\alpha \\ 0 & -D_\alpha & 0\end{pmatrix}. \tag{4}$$

These eigenvalue equations contain the spectral parameter t which does not enter the matrices X_α and Y_α. For the definitions of A_α, B_α etc in terms of the Ernst potential $\mathscr{E}$, the scalar electromagnetic potential Φ, and their derivatives, we refer the reader to our paper. $\Omega=\Omega(\zeta, \bar{\zeta}, t)$ can be considered as a 3×3 matrix array of three independent vector solutions of (2). A first integral of (2) is

$$\det\Omega=c(t)f^{3/2} \qquad c(t)=\text{constant}. \tag{5}$$

Because of the reality of the Einstein–Maxwell field the relation

$$\Omega^+\eta\Omega=C(t)f \qquad \eta=\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix} \tag{6}$$

holds, where $C(t)$ is a constant 3×3 matrix and $\Omega^+(\lambda)$ is to be understood as the Hermitian conjugate of $\Omega(1/\bar{\lambda})$. For $\lambda=\pm1$, the equations (2) can be integrated and the gauge freedom $\Omega\to\Omega g(t)$ can be chosen such that, e.g.

$$\Omega(1)=\begin{pmatrix} \bar{\mathscr{E}}+2\Phi\bar{\Phi} & 1 & i\Phi \\ \mathscr{E} & -1 & -i\Phi \\ -2i\bar{\Phi}f^{1/2} & 0 & f^{1/2}\end{pmatrix}. \tag{7}$$

In the case of *electrostatics*, the linear system (2) reduces to the 2×2 problem

$$\omega_{,\zeta}=(P_1+\lambda Q_1)\omega \qquad \omega_{,\bar{\zeta}}=(P_2+\lambda^{-1}Q_2)\omega \tag{8}$$

with the λ-independent 2×2 matrices

$$P_\alpha = \frac{1}{2}\begin{pmatrix} A_\alpha + iE_\alpha & 0 \\ 0 & A_\alpha - iE_\alpha \end{pmatrix} \qquad Q_\alpha = \frac{1}{2}\begin{pmatrix} 0 & A_\alpha + iE_\alpha \\ A_\alpha - iE_\alpha & 0 \end{pmatrix}. \tag{9}$$

The striking similarity of (8) with the corresponding equations in the vacuum case mirrors the Bonnor (1961) transformation between the axisymmetric stationary vacuum and electrostatic solutions. Any solution to (8), with components ρ, τ, implies the vector solution of (2) with the components $\psi = \rho^2 + \tau^2$, $\chi = 2\rho\tau$, $\sigma = i(\rho^2 - \tau^2)$, which can be taken as the first column of the Ω matrix. For the Reissner–Nordström metric,

$$\begin{aligned} &\mathscr{E} = \frac{R-m}{R+m} \qquad \Phi = \frac{e}{R+m} \qquad R = \tfrac{1}{2}(r_+ + r_-) \\ &r_\pm^2 = \rho^2 + (z \pm d)^2 \qquad d^2 = m^2 - e^2 \end{aligned} \tag{10}$$

we obtain the potentials

$$\begin{aligned} &\psi = \frac{1}{d^2(R+m)^2}[M_+(mR+d^2) - e^2(R^2-d^2)] \\ &\chi = \frac{1}{d(R+m)}M_- \qquad \sigma = \frac{ie(R^2-d^2)^{1/2}}{d^2(R+m)^2}[M_+ - (mR+d^2)] \end{aligned} \tag{11}$$

$$M_\pm = \tfrac{1}{2}(m+d)(R+d)\left(\frac{\lambda-\lambda_2}{\lambda-\lambda_1}\right)\left(\frac{1+\lambda_1}{1+\lambda_2}\right) \pm \tfrac{1}{2}(m-d)(R-d)\left(\frac{\lambda-\lambda_1}{\lambda-\lambda_2}\right)\left(\frac{1+\lambda_2}{1+\lambda_1}\right).$$

λ_1, λ_2 are the special values of λ for $t = \pm d/2$, hence

$$\left(\frac{1-\lambda_2}{1+\lambda_2}\right)\left(\frac{1+\lambda_1}{1-\lambda_1}\right) = \frac{R-d}{R+d}. \tag{12}$$

For $\lambda = 1$, the potentials ψ, χ, σ agree with the first column in (7).

3. The F potentials

In the metric (1), the Einstein–Maxwell equations can be cast into the form (Kinnersley and Chitre 1977)

$$\varphi^A{}_{,\zeta} = -\frac{1}{\rho}f^A{}_B\varphi^B{}_{,\zeta} \qquad \varphi^A{}_{,\bar\zeta} = \frac{1}{\rho}f^A{}_B\varphi^B{}_{,\bar\zeta} \tag{13}$$

$$H^A{}_{B,\zeta} = -\frac{1}{\rho}f^A{}_C H^C{}_{B,\zeta} \qquad H^A{}_{B,\bar\zeta} = \frac{1}{\rho}f^A{}_C H^C{}_{B,\bar\zeta}. \tag{14}$$

The indices are raised and lowered according to the usual rules

$$\varphi^A = \varepsilon^{AB}\varphi_B \qquad \varphi_A = \varphi^B\varepsilon_{BA} \qquad \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{15}$$

Accordingly, $f^A{}_B$ is given by the 2×2 matrix

$$f^A{}_B = \begin{pmatrix} -f\omega & f\omega^2 - f^{-1}\rho^2 \\ -f & f\omega \end{pmatrix}. \tag{16}$$

The potentials $\varphi^A = A^A + iB^A$ are complex combinations of the components A^A of the electromagnetic 4-potential and of potentials B^A defined by

$$A^A{}_{,\zeta} = -\frac{i}{\rho} f^A{}_B B^B{}_{,\zeta} \qquad B^A{}_{,\zeta} = \frac{i}{\rho} f^A{}_B A^B{}_{,\zeta}. \tag{17}$$

The potentials $H^A{}_B$ in (14) are determined by

$$H^A{}_B = f^A{}_B + i\Omega^A{}_B - \bar{\varphi}^A \varphi_B + \varepsilon^A{}_B K \tag{18}$$

$$\Omega^A{}_{B,\zeta} = \frac{i}{\rho} f^A{}_C (f^C{}_{B,\zeta} - 2A_B A^C{}_{,\zeta} - 2B_B B^C{}_{,\zeta}) \tag{19}$$

$$K_{,\zeta} = \bar{\varphi}_A \varphi^A{}_{,\zeta} \qquad K_{,\bar{\zeta}} = \bar{\varphi}_A \varphi^A{}_{,\bar{\zeta}}. \tag{20}$$

The Ernst potential $\mathscr{E}$ and the electromagnetic potential Φ are the components $\mathscr{E} = H_{11}$ and $\Phi = \varphi_1$. In our calculations we made use of the relations

$$\mathscr{E}_{,\zeta} = f_{,\zeta} + \frac{1}{\rho} f^2 \omega_{,\zeta} - 2\bar{\Phi}\Phi_{,\zeta} \tag{21}$$

$$\mathscr{E}_{,\bar{\zeta}} = f_{,\bar{\zeta}} - \frac{1}{\rho} f^2 \omega_{,\bar{\zeta}} - 2\bar{\Phi}\Phi_{,\bar{\zeta}} \tag{22}$$

$$H_{12} - H_{21} = 2iz + 2K \tag{23}$$

which are a consequence of equations (13)–(20). It is convenient to introduce the 3×3 matrix

$$H = \begin{pmatrix} H^A{}_B & \varphi^A \\ 2L_B & 2K \end{pmatrix} \tag{24}$$

where L_B is defined by

$$L_{B,\zeta} = \bar{\varphi}_A H^A{}_{B,\zeta} \qquad L_{B,\bar{\zeta}} = \bar{\varphi}_A H^A{}_{B,\bar{\zeta}}. \tag{25}$$

Arranging the F potentials (Hauser and Ernst 1980, Jones 1980, Kinnersley and Chitre 1977, 1978a, b) in the 3×3 matrix

$$F = \begin{pmatrix} F^A{}_B & F^A{}_3 \\ -iF_{3B} & -iF_{33} \end{pmatrix} \tag{26}$$

the linear equations for F read

$$F_{,\zeta} = \frac{it}{1+2it\zeta} H_{,\zeta} F \qquad F_{,\bar{\zeta}} = \frac{it}{1-2it\bar{\zeta}} H_{,\bar{\zeta}} F. \tag{27a, b}$$

Here t again denotes a complex constant parameter; H is independent of t, but $F = F(\zeta, \bar{\zeta}, t)$. The complex conjugate of $F(\bar{t})$, denoted by $F^*(t)$, satisfies the equations

$$F^*{}_{,\bar{\zeta}} = \frac{-it}{1-2it\bar{\zeta}} \overline{H_{,\zeta}} F^* \qquad F^*{}_{,\zeta} = \frac{-it}{1+2it\zeta} \overline{H_{,\bar{\zeta}}} F^*. \tag{28}$$

The integrability condition of (27) leads to the second-order equation

$$H_{,\zeta\bar{\zeta}} + \frac{1}{4\rho}(H_{,\bar{\zeta}} H_{,\zeta} - H_{,\zeta} H_{,\bar{\zeta}}) = 0 \tag{29}$$

for H. The (2, 1)- and (2, 3)-components of this relation are just the field equations for $\mathscr{E}$ and Φ. This can be checked using equations (13)–(25).

4. The relationship between the F and Ω potentials

A straightforward calculation yields the explicit form

$$H_{,\zeta} = -\begin{pmatrix} p\mathscr{E}_{,\zeta} & p(1-p\mathscr{E}_{,\zeta}-q\Phi_{,\zeta}) & p\Phi_{,\zeta} \\ \mathscr{E}_{,\zeta} & 1-p\mathscr{E}_{,\zeta}-q\Phi_{,\zeta} & \Phi_{,\zeta} \\ q\mathscr{E}_{,\zeta} & q(1-p\mathscr{E}_{,\zeta}-q\Phi_{,\zeta}) & q\Phi_{,\zeta} \end{pmatrix} \tag{30}$$

of the matrix $H_{,\zeta}$ in (27), where the abbreviations

$$p = f^{-1}\rho + \omega \qquad q = -2(p\bar{\Phi} + \bar{\varphi}_2) \tag{31}$$

have been used. Clearly, one has $\det(H_{,\zeta}) = 0$.

With (30), the equations (27a) take the form

$$\begin{aligned} &F^1{}_{1,\zeta} = pF^2{}_{1,\zeta} \qquad F^3{}_{1,\zeta} = qF^2{}_{1,\zeta} \\ &F^2{}_{1,\zeta} = \frac{-\mathrm{i}t}{1+2\mathrm{i}t\zeta}[\mathscr{E}_{,\zeta}F^1{}_1 + (1-p\mathscr{E}_{,\zeta}-q\Phi_{,\zeta})F^2{}_1 + \Phi_{,\zeta}F^3{}_1]. \end{aligned} \tag{32}$$

In terms of the new quantities

$$R_1 = F^1{}_1 - pF^2{}_1 \qquad R_2 = F^2{}_1 \qquad R_3 = F^3{}_1 - qF^2{}_1 \tag{33}$$

the equations (32) read

$$\begin{aligned} &R_{1,\zeta} = -p_{,\zeta}R_2 \qquad R_{3,\zeta} = -q_{,\zeta}R_2 \\ &R_{2,\zeta} = -\frac{\mathrm{i}t}{1+2\mathrm{i}t\zeta}(\mathscr{E}_{,\zeta}R_1 + R_2 + \Phi_{,\zeta}R_3). \end{aligned} \tag{34}$$

This form of the equations enables us to read off the generalisation of the known relation between Ω and F in the vacuum case (Cosgrove 1980). Equation (27b) can be treated in the same manner. The final result of our calculations is the transformation

$$\Omega = \begin{pmatrix} 2\mathrm{i}tf & 1-2tz-2\mathrm{i}tf\omega & 0 \\ 0 & -S & 0 \\ 2t\bar{\Phi}f^{1/2} & 2t\bar{\varphi}_2 f^{1/2} & -tf^{1/2} \end{pmatrix} Fg(t) \tag{35}$$

where

$$S = (1+2\mathrm{i}t\zeta)^{1/2}(1-2\mathrm{i}t\bar{\zeta})^{1/2} \tag{36}$$

and $g(t)$ is a constant gauge matrix. Thus we have shown the equivalence of the linear systems (2) and (27). Note that the transformation matrix in (35) includes metric functions, electromagnetic potentials, and the spectral parameter t.

The transformation (35) maps the equations (5), (6) into the corresponding relations for the F potentials (see Hauser and Ernst 1980). The Ω potentials (11) of the Reissner–Nordström metric are related by (35) to the F potentials given by Jones (1980).

5. The Ψ potentials

Starting with the eigenvalue equations

$$\Psi_{,a} = \Lambda_a{}^b U_{,b} \Psi \qquad x^a = (\rho, z)$$
$$\Lambda_a{}^b = \frac{\mathrm{i}}{2} \frac{(z-w)\delta_a{}^b + \rho\epsilon_a{}^b}{\rho^2 + (z-w)^2} \qquad \epsilon_a{}^b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{37}$$

w being the constant spectral parameter, Aleksejev (1980a) constructed N-soliton solutions of the Einstein–Maxwell equations. The relationship between the linear systems (27) and (37) is given by

$$H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} U \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Psi G(t) \qquad w = \frac{1}{2t}. \tag{38}$$

From (24) and (38) it follows that U has the form

$$U = \begin{pmatrix} H_A{}^B & \varphi_A \\ -2L^B & -2K \end{pmatrix}. \tag{39}$$

With the aid of the transformations (35) and (38) it can be shown that Aleksejev's equation

$$\Psi^+ W \Psi = K(t) = \text{constant} \tag{40}$$
$$W = \begin{pmatrix} g^{AB} + 4\varphi^A \bar{\varphi}^B & -2\varphi^A \\ -2\bar{\varphi}^B & 1 \end{pmatrix} \qquad g^{AB} = -4f^{AB} + 4\mathrm{i}\left(\frac{1}{2t} - z\right)\varepsilon^{AB}$$

is equivalent with (6). To generate new solutions (represented by Ψ') from an initial solution (Ψ), Aleksejev (1980a) assumed the special structure

$$\Psi' = \chi \Psi \qquad \chi = I + \sum_{l=1}^{N} \frac{R_l}{w - w_l} \tag{41}$$

where the 3×3 matrices R_l are independent of w. In terms of the F potentials, the ansatz (41) leads, for $N = 1$, to the formula

$$F'(t) = \left(I + \frac{t(s-\bar{s})}{\bar{s}(t-s)} \cdot \frac{F(\bar{s}) h \otimes g F^{-1}(s)}{(g F^{-1}(s) F(\bar{s}) h)} \right) F(t) \tag{42}$$

which turns out to be identical (up to gauge transformation) with the transformation found by Cosgrove (1981). The value in the complex t plane at which χ in (41) has a simple pole is denoted by s. (χ^{-1} has a pole at $t = \bar{s}$.) The symbol $\otimes$ means dyadic product. The constant row and column vectors g and h, respectively, are related by

$$g = kG^{-1}(s) \qquad h = G(\bar{s}) K^{-1}(\bar{s}) k^+ \tag{43}$$

k being a constant row vector. In particular, the choice

$$G(t) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad K^{-1}(t) = \frac{t}{2\mathrm{i}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2\mathrm{i}/t \end{pmatrix} \tag{44}$$

(special HE gauge, see Cosgrove 1981) implies

$$h_1 = \tfrac{1}{2}\mathrm{i}\bar{s}\bar{g}_2 \qquad h_2 = -\tfrac{1}{2}\mathrm{i}\bar{s}\bar{g}_1 \qquad h_3 = \bar{g}_3. \tag{45}$$

From (27) and (42) one obtains the H matrix

$$H' = H + \frac{\mathrm{i}(s-\bar{s})}{s\bar{s}} \cdot \frac{F(\bar{s})h \otimes gF^{-1}(s)}{(gF^{-1}(s)F(\bar{s})h)} \tag{46}$$

of the new solution. Cosgrove (1981) derived the same result using the Hauser–Ernst (1980) formulation in terms of a homogeneous Hilbert problem.

In the modified version of the N-soliton solution (Aleksejev 1980b), the degenerate matrices R_l in (41) have the form

$$R_l = n_l \otimes m_l + r_l \otimes s_l. \tag{47}$$

For $N = 1$, the corresponding transformation, when applied to flat space–time, does not give a more general solution; like (46) it leads again to the Kerr–Newman solution.

6. Summary

The linear equations (2), (27), (37) for the (pseudo-) potentials Ω, F and Ψ, respectively, are equivalent. The various formulations are related by the transformations (35) and (38).

Acknowledgment

I wish to thank Dr G Neugebauer and Dr C Hoenselaers for stimulating discussions.

References

Aleksejev G A 1980a *Pis'ma Zh. Eksp. Teor. Fiz.* **32** 301
——1980b *Abstracts GR9, Jena* vol 1, p 1
Belinsky V A and Zakharov V E 1978 *Zh. Eksp. Teor. Fiz.* **75** 1953 (*Sov. Phys.-JETP* **48** 985)
——1979 *Zh. Eksp. Teor. Fiz.* **77** 3 (*Sov. Phys.-JETP* **50** 1)
Bonnor W B 1961 *Z. Phys.* **161** 439
Cosgrove C M 1980 *J. Math. Phys.* **21** 2417
——1981 *California Institute of Technology Preprint* No OAP-619
Harrison B K 1978 *Phys. Rev. Lett.* **41** 1197
Hauser I and Ernst F J 1979 *Phys. Rev.* D **20** 362, 1783
——1980 *J. Math. Phys.* **21** 1126, 1418
Jones T C 1980 *J. Math. Phys.* **21** 1790
Kinnersley W 1977 *J. Math. Phys.* **18** 1529
Kinnersley W and Chitre D M 1977 *J. Math. Phys.* **18** 1538
——1978a *J. Math. Phys.* **19** 1926, 2037
——1978b *Phys. Rev. Lett.* **40** 1608
Kramer D and Neugebauer G 1981 *J. Phys. A: Math. Gen.* **14** L333
Neugebauer G 1979 *J. Phys. A: Math. Gen.* **12** L67
——1980 *J. Phys. A: Math. Gen.* **13** L19, 1737
Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1